

Asymptotic pricing in large financial markets

Michał Barski

Faculty of Mathematics and Computer Science, University of Leipzig, Germany
 Faculty of Mathematics, Cardinal Stefan Wyszyński University in Warsaw, Poland
Michal.Barski@math.uni-leipzig.de

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Abstract

The problem of hedging and pricing sequences of contingent claims in large financial markets is studied. Connection between asymptotic arbitrage and behavior of the α - quantile price is shown. The large Black-Scholes model is carefully examined.

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1 Introduction

A large financial market is a sequence of small arbitrage-free markets. Absence of arbitrage opportunity on each element of the sequence does not guarantee that there is no arbitrage "in the limit". Different concepts of asymptotic arbitrage were introduced in [7] and [8] and their connections with some properties of measure families: contiguity and asymptotic separation were shown. For other similar results in this field see also [9], [10]. For other notions as asymptotic free lunch and its relation with existence of a martingale measure for the whole market see [9], [12].

Another problem arises as a natural consequence of asymptotic arbitrage theory: how one can calculate the price of a contingent claim and what is the connection between the price and the no-arbitrage property of the market. We formulate the problem of pricing not for a single random variable but for the sequence of random variables instead. Motivation for such problem stating is presented in section 3. For such a sequence we define different types of sequences of hedging strategies. The first of them hedges each element of the sequence, thus it carries no risk at all. Basing on that property we define a strong price, which is strictly related to the price known from the classical theory of financial markets. The other type hedges the sequence with some risk which does not exceed a fixed level in infinity. For this case we introduce the α -quantile price. In particular, the risk can vanish in infinity indicating the 1-quantile price which is called a weak price. These definitions are presented in section 3.

In section 4 we provide characterization theorems for the prices mentioned above for general large financial markets. This general description uses the no-arbitrage property of each small market only. The question arises how the prices are related to each other, in particular the strong and the weak one, under different types of asymptotic arbitrage. Example 4.6 shows that asymptotic arbitrage actually does affect this relation. We study this problem and show a relevant theorem for the sequence of complete markets. Analogous theorem for incomplete

markets remains an open problem.

A significant part of the paper is section 5 devoted to the large Black-Scholes market with constant coefficients. In these particular settings we improved previous results and established more precise characterization theorems which includes widely used derivatives such as call and put options. In this section we also provide an alternative proof of the theorem describing different kinds of asymptotic arbitrage which comes from [8]. The method of proving is less general than in [8], but using Neyman-Pearson lemma provides more indirect insight into the construction of relevant sets. Moreover, similar methods based on non-randomized tests are successfully used in other proofs in this section.

The paper is organized as follows. In section 2 we present definitions of some properties of measure families and known facts about asymptotic arbitrage. For a more comprehensive exposition see [5] for the statistical part and [7], [8], [9], [10] for the financial part. In section 3 we formulate precisely the problem of pricing. Section 4 provides characterization theorems which are used and generalized in section 5 describing the large Black-Scholes model.

In general, the main idea in the α -quantile price characterization theorem has its origin in the paper on quantile hedging [4]. Thus the results presented here can be treated as an extension or further development in this field.

2 Basic definitions and results

By a large financial market we mean a sequence of small markets. Let $(\Omega^n, \mathcal{F}^n, (\mathcal{F}_t^n), P^n)$, where $t \in [0, T^n]$ or $t \in \{0, 1, \dots, T^n\}$ be a sequence of filtered probability spaces and $(S_n^i(t))$, $i = 1, 2, \dots, d_n$ a sequence of semimartingales describing evolution of d_n stock prices. A large financial market will be called **stationary** if $S_{n+1}^i(t) = S_n^i$ for $i = 1, 2, \dots, d_n$. This means that each subsequent small market contains the previous one. To shorten notation assume that all the markets have the same time horizon, i.e. $T^n = T$ for $n = 1, 2, \dots$.

As a trading strategy on the n -th small market we admit a pair (x_n, φ_n) , where $x_n \geq 0$ and φ_n is an \mathbb{R}^{d_n} valued predictable process integrable with respect to $(S_n(t))$. The value of x_n is an initial endowment and $\varphi_n^i(t)$ is a number of units of the i -th stock held in the portfolio at time t . The wealth process corresponding to the strategy (x_n, φ_n) defined as $V_t^{x_n, \varphi_n} = \sum_{i=1}^{d_n} \varphi_n^i(t) S_n^i(t)$ is assumed to satisfy a self-financing condition, that is:

$$V_t^{x_n, \varphi_n} = x_n + \int_0^t \varphi_n(s) dS_n(s) \text{ for continuous time models}$$

$$V_t^{x_n, \varphi_n} = x_n + \sum_{s=1}^t \sum_{i=1}^{d_n} \varphi_n^i(s) (S_n^i(s) - S_n^i(s-1)) \text{ for discrete time models.}$$

Definition 2.1 A pair $(0, \varphi_n)$ is an arbitrage strategy on the n -th small market if $V_t^{0, \varphi_n} \geq 0$ a.s. for each t and

$$P^n(V_T^{0, \varphi_n} > 0) > 0.$$

For the n -th small market we recall the definition of the set \mathcal{Q}^n of all martingale measures.

Definition 2.2 $Q \in \mathcal{Q}^n \iff (S_n^i(t))$ is a local martingale on $[0, T]$ with respect to Q for $i = 1, 2, \dots, d_n$

Theorem 2.3 If $\mathcal{Q}^n \neq \emptyset$ then there is no arbitrage strategy on the n -th small market.

The proof can be found in [5] for discrete time and in [1] for continuous time settings. It turns out that the inverse statement remains true for the discrete case, but is false for continuous time.

Throughout all the paper we assume that:

$$\mathcal{Q}^n \neq \emptyset \quad \text{for all } n = 1, 2, \dots$$

The fact that there is no arbitrage on each small market does not guarantee that there is no asymptotic arbitrage opportunity. For the large financial markets we have the following concepts of asymptotic arbitrage which comes from [8].

Definition 2.4 *A sequence of strategies (x_n, φ_n) realizes the asymptotic arbitrage of the first kind (AA1) if:*

$$\begin{aligned} V_t^{x_n, \varphi_n} &\geq 0 \text{ for all } t \in [0, T] \\ \lim_n x_n &= 0, \\ \lim_n P^n(V_T^{x_n, \varphi_n} \geq 1) &> 0. \end{aligned}$$

Definition 2.5 *A sequence of strategies (x_n, φ_n) realizes the asymptotic arbitrage of the second kind (AA2) if:*

$$\begin{aligned} V_t^{x_n, \varphi_n} &\leq 1 \text{ for all } t \in [0, T] \\ \lim_n x_n &> 0, \\ \lim_n P^n(V_T^{x_n, \varphi_n} \geq \varepsilon) &= 0 \text{ for any } \varepsilon > 0. \end{aligned}$$

Definition 2.6 *A sequence of strategies (x_n, φ_n) realizes the strong asymptotic arbitrage of the first kind (SAA1) if:*

$$\begin{aligned} V_t^{x_n, \varphi_n} &\geq 0 \text{ for all } t \in [0, T] \\ \lim_n x_n &= 0, \\ \lim_n P^n(V_T^{x_n, \varphi_n} \geq 1) &= 1. \end{aligned}$$

Definition 2.7 *A sequence of strategies (x_n, φ_n) realizes the strong asymptotic arbitrage of the second kind (SAA2) if:*

$$\begin{aligned} V_t^{x_n, \varphi_n} &\leq 1 \text{ for all } t \in [0, T] \\ \lim_n x_n &= 1, \\ P^n(V_T^{x_n, \varphi_n} \geq \varepsilon) &= 0 \text{ for any } \varepsilon > 0. \end{aligned}$$

We say that the large financial market does not admit the asymptotic arbitrage of the first kind (second kind, strong asymptotic arbitrage of the first kind, strong asymptotic arbitrage of the second kind) and denote this property by $NAA1$, ($NAA2$, $NSAA1$, $NSAA2$) if for any sequence (n_k) there are no trading strategies (x_{n_k}, φ_{n_k}) realizing the corresponding kind of asymptotic arbitrage.

For characterization of the asymptotic arbitrage and for later purposes we introduce some definitions from mathematical statistics.

Definition 2.8 Let $(\Omega^n, \mathcal{F}^n), n = 1, 2, \dots$ be a sequence of measurable spaces and $G_n, H_n : \mathcal{F}^n \rightarrow \mathbb{R}_+$ a sequence of set functions.

1) (G_n) is **contiguous** with respect to (H_n) (notation: $(G_n) \triangleleft (H_n)$) if for every sequence $A_n \in \mathcal{F}^n$ we have

$$H_n(A_n) \rightarrow 0 \implies G_n(A_n) \rightarrow 0$$

2) (G_n) is **asymptotically separable** from (H_n) (notation: $(G_n) \triangle (H_n)$) if there exists a sequence $A_n \in \mathcal{F}^n$ such that

$$H_n(A_n) \rightarrow 0 \quad \text{and} \quad G_n(A_n) \rightarrow 1$$

For the family \mathcal{Q}^n we consider the following set functions:

$$\begin{aligned} \bar{\mathbf{Q}}^n(A) &= \sup_{Q \in \mathcal{Q}^n} Q(A), \quad A \in \mathcal{F}^n \quad - \text{the upper envelope of } \mathcal{Q}^n \\ \underline{\mathbf{Q}}^n(A) &= \inf_{Q \in \mathcal{Q}^n} Q(A), \quad A \in \mathcal{F}^n \quad - \text{the lower envelope of } \mathcal{Q}^n. \end{aligned}$$

The following result provides characterization of asymptotic arbitrage in terms of sequences of sets. For the proofs see [7], [8], [10].

Theorem 2.9 *The following conditions hold*

1. (NAA1) iff $(P^n) \triangleleft (\bar{\mathbf{Q}}^n)$
2. (NAA2) iff $(\underline{\mathbf{Q}}^n) \triangleleft (P^n)$
3. (SAA1) iff (SAA2) iff $(P^n) \triangle (\bar{\mathbf{Q}}^n)$ iff $(\underline{\mathbf{Q}}^n) \triangle (P^n)$.

Below we present a standard tool from mathematical statistics for searching optimal tests. It is useful to solve the following problem. Let Q_1 and Q_2 be two probability measures with density $\frac{dQ_1}{dQ_2}$ on a measurable space (Ω, \mathcal{F}) . We are interested in finding set \tilde{A} , which is a solution of the problem

$$A \in \mathcal{F} : \begin{cases} Q_1(A) \rightarrow \max \\ Q_2(A) \leq \gamma \end{cases}$$

with $\gamma \in [0, 1]$. Then the explicit solution is given by the following lemma.

Lemma 2.10 (Neyman-Pearson)

If there exists constant β such that $Q_2\{\frac{dQ_1}{dQ_2} \geq \beta\} = \gamma$ then $Q_1\{\frac{dQ_1}{dQ_2} \geq \beta\} \geq Q_1(B)$ for any set B satisfying $Q_2(B) \leq \gamma$.

We recall also the pricing theorem, which has its origin in the theorem on optional decomposition of the supermartingales. For more details see [11] and for later extensions [2], [3].

Theorem 2.11 (Price characterization) Let \mathcal{Q} be a set of martingale measures for the semimartingale (S_t) describing evolution of the stock prices. Let H be a non negative contingent claim. Then there exists a trading strategy $(\tilde{x}, \tilde{\varphi})$, where $\tilde{x} = \sup_{Q \in \mathcal{Q}} \mathbf{E}^Q[H]$ s.t.

$$\tilde{x} + \int_0^t \tilde{\varphi}(s) dS(s) \geq \text{ess sup}_{Q \in \mathcal{Q}} \mathbf{E}^Q[H \mid \mathcal{F}_t].$$

The pair $(\tilde{x}, \tilde{\varphi})$ is thus a hedging strategy and \tilde{x} is the price of H .

3 Problem formulation

Definition 3.1 A contingent claim \mathbb{H} on a large financial market is a sequence of random variables H_1, H_2, \dots satisfying the following conditions

- 1) For each $n = 1, 2, \dots$ $H_n : \Omega^n \rightarrow \mathbb{R}^+$ is an \mathcal{F}^n measurable, non negative random variable.
- 2) For each $n = 1, 2, \dots$ $\sup_{Q \in \mathcal{Q}^n} \mathbf{E}^Q[H_n] < \infty$ holds.

In classical market models we have always one random variable which we want to price and hedge. The question arises for justification of considering a sequence of random variables. We present two motivations for this fact.

- 1) Assume that we have one random variable G which is measurable with respect to the σ -field $\sigma(\mathcal{F}^1, \mathcal{F}^2, \dots)$. Then H_n can be defined as projections of G on the spaces $(\Omega^n, \mathcal{F}^n, P^n)$, i.e. $H_n = \mathbf{E}^{P^n}[G \mid \mathcal{F}^n]$. Thus, we want to price a derivative which depends on infinitely many assets but taking into account information which is provided by the few coming first.
- 2) Let G be a random variable which depends on the price of the first asset (or some first assets as well) only. Then we can define $H_n = G$ for each n and consider opportunity arising from the fact that the number of assets which can be traded is increasing. We examine how the increasing number of investments possibilities affects the price of G .

Below we present two concepts of asymptotic hedging and prices definitions of \mathbb{H} .

Definition 3.2 A sequence $(x_n, \varphi_n)_n$ is a **sequence of hedging strategies** if

$$V_T^{x_n, \varphi_n} \geq H_n \quad \forall n = 1, 2, \dots$$

Such class of sequences we denote by \mathcal{H}^1 . A **strong price** of \mathbb{H} is defined as

$$v(\mathbb{H}) = \inf_{(x_n, \varphi_n) \in \mathcal{H}^1} \lim_{n \rightarrow \infty} x_n.$$

Throughout the whole paper we assume that α is any number from the interval $[0, 1]$.

Definition 3.3 A sequence $(x_n, \varphi_n)_n$ is a **sequence of α -hedging strategies** if

$$\lim_{n \rightarrow \infty} P^n(V_T^{x_n, \varphi_n} \geq H_n) \geq \alpha.$$

Such class of sequences we denote by \mathcal{H}_α . An **α -quantile price** of \mathbb{H} is defined as

$$v_\alpha(\mathbb{H}) = \inf_{(x_n, \varphi_n) \in \mathcal{H}_\alpha} \lim_{n \rightarrow \infty} x_n.$$

A **weak price** of \mathbb{H} is the 1-quantile price, i.e.

$$\tilde{v}(\mathbb{H}) := v_1(\mathbb{H}).$$

As follows from the definition above, we consider sequences of strategies which do not allow to exceed a fixed level of risk when n tends to infinity. If $\alpha = 1$, then the risk vanishes in infinity. This particular case is distinguished to compare with classical concept of pricing suggested by Definition 3.2, where there is no risk for any $n = 1, 2, \dots$.

At this stage it is clear that $v_\alpha(\mathbb{H}) \leq v_\beta(\mathbb{H}) \leq \tilde{v}(\mathbb{H}) \leq v(\mathbb{H})$ for $\alpha < \beta$ since the following inclusions hold : $\mathcal{H}_\alpha \supseteq \mathcal{H}_\beta \supseteq \mathcal{H}_1 \supseteq \mathcal{H}^1$. The main goal of the paper is to provide the characterization of the prices and solve the problem of equality between the strong and the weak price.

4 Prices characterization

Using the price characterization Theorem 2.11 on a classical market it is simple to show the following.

Proposition 4.1 *The strong price is given by*

$$v(\mathbb{H}) = \lim_n \sup_{Q \in \mathcal{Q}^n} \mathbf{E}^Q[H_n].$$

Proof : Let $g := \lim_n \sup_{Q \in \mathcal{Q}^n} \mathbf{E}^Q[H_n]$. By Theorem 2.11, for any $(x_n, \varphi_n) \in \mathcal{H}^1$ we get $x_n \geq \sup_{Q \in \mathcal{Q}^n} \mathbf{E}^Q[H_n]$ and thus $v(\mathbb{H}) \geq g$.
Taking $\tilde{x}_n := \sup_{Q \in \mathcal{Q}^n} \mathbf{E}^Q[H_n]$, from Theorem 2.11 we know that there exists a sequence of strategies $(\tilde{\varphi}_n)$ s.t. $(\tilde{x}_n, \tilde{\varphi}_n) \in \mathcal{H}^1$ and thus $v(\mathbb{H}) \leq g$. \square

To characterize the weak price we introduce first some definitions.

Definition 4.2 *(The class \mathcal{A}_α)*

A sequence of sets (A_n) belongs to the class \mathcal{A}_α if $A_n \in \mathcal{F}^n$ for $n = 1, 2, \dots$ and $\lim_{n \rightarrow \infty} P^n(A_n) \geq \alpha$. In particular (A_n) belongs to the class \mathcal{A}_1 if $P^n(A_n) \xrightarrow[n]{} 1$.

The following remarks state the correspondence between the class of α -hedging sequences \mathcal{H}_α and the class of \mathcal{A}_α sets.

Remark 4.3 *($\mathcal{H}_{\mathcal{A}_\alpha} \subseteq \mathcal{H}_\alpha$)*

Each element in \mathcal{H}_α indicates an element in \mathcal{A}_α . Indeed, for $(x_n, \varphi_n) \in \mathcal{H}_\alpha$ let us define $A_n^{x_n, \varphi_n} := \{V_T^{x_n, \varphi_n} \geq H_n\}$. By definition of \mathcal{H}_α we obtain that $(A_n^{x_n, \varphi_n}) \in \mathcal{A}_\alpha$. Thus, if we denote the sequences of sets above by $\mathcal{A}_{\mathcal{H}_\alpha}$ the following inclusion holds : $\mathcal{A}_{\mathcal{H}_\alpha} \subseteq \mathcal{A}$.

Remark 4.4 *($\mathcal{H}_{\mathcal{A}_\alpha} \subseteq \mathcal{H}_\alpha$)*

For the sequence $(A_n) \in \mathcal{A}_\alpha$ let us consider a sequence of strategies s.t. for a fixed number n strategy (x_n^A, φ_n^A) satisfies : $x_n^A = \sup_{Q \in \mathcal{Q}^n} \mathbf{E}^Q[H_n \mathbf{1}_{A_n}]$ and φ_n^A hedges the contingent claim $H_n \mathbf{1}_{A_n}$ (on a small market with index n). It follows that $(x_n^A, \varphi_n^A) \in \mathcal{H}_\alpha$ since $(A_n) \in \mathcal{A}_\alpha$. If we denote the sequences of strategies of the form above by $\mathcal{H}_{\mathcal{A}_\alpha}$ the following inclusion holds: $\mathcal{H}_{\mathcal{A}_\alpha} \subseteq \mathcal{H}_\alpha$.

Theorem 4.5 *The α -quantile price is given by*

$$v_\alpha(\mathbb{H}) = \inf_{(A_n) \in \mathcal{A}_\alpha} \lim_n \sup_{Q \in \mathcal{Q}^n} \mathbf{E}^Q[H_n \mathbf{1}_{A_n}].$$

Proof : We show successively two inequalities: (\geq) and (\leq) .

(\geq) Let us consider $(x_n, \varphi_n) \in \mathcal{H}_\alpha$. Then using the notation of Remark 4.3 we have:

$$x_n \geq \sup_{Q \in \mathcal{Q}^n} \mathbf{E}^Q[H_n \mathbf{1}_{A_n^{x_n, \varphi_n}}]$$

and therefore

$$\lim_n x_n \geq \lim_n \sup_{Q \in \mathcal{Q}^n} \mathbf{E}^Q[H_n \mathbf{1}_{A_n^{x_n, \varphi_n}}].$$

By the definition of the α -quantile price and by Remark 4.3 we obtain

$$\begin{aligned}
v_\alpha(\mathbb{H}) &= \inf_{(x_n, \varphi_n) \in \mathcal{H}_\alpha} \lim_n x_n \geq \inf_{(x_n, \varphi_n) \in \mathcal{H}_\alpha} \lim_n \sup_{Q \in \mathcal{Q}^n} \mathbf{E}^Q[H_n \mathbf{1}_{A_n^{x_n, \varphi_n}}] \\
&= \inf_{(A_n) \in \mathcal{A}_{\mathcal{H}_\alpha}} \lim_n \sup_{Q \in \mathcal{Q}^n} \mathbf{E}^Q[H_n \mathbf{1}_{A_n}] \\
&\geq \inf_{(A_n) \in \mathcal{A}_\alpha} \lim_n \sup_{Q \in \mathcal{Q}^n} \mathbf{E}^Q[H_n \mathbf{1}_{A_n}]
\end{aligned}$$

(\leq) Consider an arbitrary element $(A_n) \in \mathcal{A}_\alpha$ and a corresponding strategy described in Remark 4.4. Following the notation of Remark 4.4 we have

$$\sup_{Q \in \mathcal{Q}^n} \mathbf{E}^Q[H_n \mathbf{1}_{A_n}] = x_n^A$$

and therefore

$$\lim_n \sup_{Q \in \mathcal{Q}^n} \mathbf{E}^Q[H_n \mathbf{1}_{A_n}] = \lim_n x_n^A$$

By Remark 4.4 we obtain

$$\begin{aligned}
\inf_{(A_n) \in \mathcal{A}_\alpha} \lim_n \sup_{Q \in \mathcal{Q}^n} \mathbf{E}^Q[H_n \mathbf{1}_{A_n}] &= \inf_{(A_n) \in \mathcal{A}_\alpha} \lim_n x_n^A \\
&= \inf_{(x_n, \varphi_n) \in \mathcal{H}_{\mathcal{A}_\alpha}} \lim_n x_n \\
&\geq \inf_{(x_n, \varphi_n) \in \mathcal{H}_\alpha} \lim_n x_n \\
&= v_\alpha(\mathbb{H})
\end{aligned}$$

□

We examine the problem of asymptotic pricing studying the following example.

Example 4.6 *Let us consider the stationary large financial market with the following settings:*

$$\Omega = [0, 1], \quad S_n^i(1) = S_n^i(0)(1 + \xi_i), \quad i = 1, 2, \dots, n, \quad n = 1, 2, \dots$$

where (ξ_i) is a sequence of random variables given by

$$\xi_i = \begin{cases} -1 & \text{on } [0, 1 - \frac{1}{2^i}] := E_i \\ \frac{\delta(2^i - 1)}{2^i - \delta(2^i - 1)} & \text{on } (1 - \frac{1}{2^i}, 1] := F_i, \quad \delta \in (0, 1). \end{cases}$$

Sigma fields are assumed to be generated by the sequence (ξ_i) , i.e. $\mathcal{F}^n = \sigma(\xi_1, \xi_2, \dots, \xi_n)$, and the n -th objective probability measure P^n is a restriction of the Lebesgue's measure P on $[0, 1]$ to the sigma-field \mathcal{F}^n , i.e. $P^n = P|_{\mathcal{F}^n}$. Each martingale measure Q^n on the n -th market is described by the property : $\mathbf{E}^{Q^n}[\xi_1] = 0, \mathbf{E}^{Q^n}[\xi_2] = 0, \dots, \mathbf{E}^{Q^n}[\xi_n] = 0$. Thus Q^n is indicated by its values on the intervals E_1, E_2, \dots, E_n and one can check that

$$\begin{aligned}
Q^n(E_1) &= \delta \left(1 - \frac{1}{2}\right) \\
Q^n(E_2) &= \delta \left(1 - \frac{1}{2^2}\right) \\
&\dots \\
Q^n(E_n) &= \delta \left(1 - \frac{1}{2^n}\right).
\end{aligned}$$

It follows from the above that we have constructed a sequence of complete markets. We shall find an α -quantile price of a trivial contingent claim $\mathbb{H} \equiv 1$.

Proposition 4.7 *In the model specified above we have:*

$$v_\alpha(1) = \delta\alpha.$$

Proof : We shall construct explicitly a sequence of sets $(\tilde{A}_n) \in \mathcal{A}_\alpha$ satisfying:

$$\lim Q^n(\tilde{A}_n) = \inf_{(A_n) \in \mathcal{A}_\alpha} \underline{\lim} Q^n(A_n).$$

Let: $G_1 := E_1$, $G_n := E_n \setminus E_{n-1}$ for $n=2,3,\dots$. Then

$$P^n(G_n) = P^n(F_n) = \frac{1}{2^n} \quad \text{and} \\ \frac{\delta}{2^n} = Q^n(G_n) < Q^n(F_n) = 1 - \delta \left(1 - \frac{1}{2^n}\right)$$

Consider a series expansion of α :

$$\alpha = \sum_{i=1}^{\infty} \frac{\gamma_i}{2^i}, \quad \text{where } \gamma_i \in \{0, 1\}.$$

Define \tilde{A}_n as follows

$$\tilde{A}_n := \bigcup_{i=1}^n \mathbf{1}_{\{\gamma_i=1\}} G_i$$

and notice, that $P^n(\tilde{A}_n) = \sum_{i=1}^n \gamma_i P^n(G_i) = \sum_{i=1}^n \frac{\gamma_i}{2^i}$ and therefore $\lim_{n \rightarrow \infty} P^n(\tilde{A}_n) = \sum_{i=1}^{\infty} \frac{\gamma_i}{2^i} = \alpha$, so $(\tilde{A}_n) \in \mathcal{A}_\alpha$. For any $(A_n) \in \mathcal{A}_\alpha$ we have $\lim P^n(\tilde{A}_n) \leq \underline{\lim} P^n(A_n)$ and $\lim Q^n(\tilde{A}_n) \leq \underline{\lim} Q^n(A_n)$.

Thus

$$\begin{aligned} v_\alpha(1) &= \inf_{(A_n) \in \mathcal{A}_\alpha} \underline{\lim}_n \mathbf{E}^{Q^n}[\mathbf{1}_{A_n}] = \lim_n Q^n[\tilde{A}_n] = \lim_n \sum_{i=1}^n \mathbf{1}_{\{\gamma_i=1\}} Q^n(G_i) \\ &= \sum_{i=1}^{\infty} \gamma_i \frac{\delta}{2^i} = \delta \sum_{i=1}^{\infty} \frac{\gamma_i}{2^i} = \delta\alpha. \end{aligned}$$

□

Notice that $\delta = \tilde{v}(1) < v(1) = \underline{\lim}_n \mathbf{E}^{Q^n}[1] = 1$, so this example shows that strict inequality between the strong and the weak price is possible.

Notice also that this model admits AA2 and does not satisfy AA1. Indeed, taking the sequence (F_n) , we get: $P^n(F_n) = \frac{1}{2^n} \rightarrow 0$ and $Q^n(F_n) = 1 - \delta \left(1 - \frac{1}{2^n}\right) \rightarrow 1 - \delta > 0$ and thus there is AA2. Let (A_n) be a sequence s.t. $Q^n(A_n) \rightarrow 0$. This means that for any $l > 0$, $Q^n(A_n) < \frac{\delta}{2^l}$ holds for all large n and one can check, that this implies that $A_n \subseteq (1 - \frac{1}{2^l}, 1]$ for all large n . As a consequence we obtain $\lim_n P^n(A_n) < \frac{1}{2^l}$ and letting l to ∞ we get $\lim_n P^n(A_n) = 0$. This means that NAA1 and also NSAA1, NSAA2 hold.

This example shows that NAA1, NSAA1, NSAA2 is insufficient for the equality of the strong and the weak price.

Theorem 4.5 yields immediately two following conclusions.

Remark 4.8 *If we require that $\tilde{v}(\mathbb{H}) = v(\mathbb{H})$ even for \mathbb{H} of simple structure then the market must satisfy NAA2. Indeed, suppose that AA2 holds. It implies that for any $(A_n) \in \mathcal{A}_1$, $\bar{Q}^n(A_n) \rightarrow 1$ holds. Taking $\mathbb{H} \equiv 1$ we obtain*

$$\tilde{v}(1) = \inf_{(A_n) \in \mathcal{A}_1} \lim_n \bar{Q}^n(A_n) < 1 = v(1).$$

Remark 4.9 *If there is SAA1 or equivalently SAA2, then for any \mathbb{H} bounded, i.e. $H_n \leq K$ for some constant $K > 0$, we have $v_\alpha(\mathbb{H}) = 0$ for any $\alpha \in [0, 1]$. Indeed, by Theorem 2.9 there exists a sequence (\tilde{A}_n) s.t. $P^n(\tilde{A}_n) \rightarrow 1$ and $\bar{Q}^n(\tilde{A}_n) \rightarrow 0$. Then we have*

$$v_\alpha(\mathbb{H}) \leq \tilde{v}(\mathbb{H}) \leq \lim_n \sup_{Q \in \mathcal{Q}^n} \mathbf{E}^{Q^n}[H_n \mathbf{1}_{\tilde{A}_n}] \leq \lim_n K \bar{Q}^n(\tilde{A}_n) = 0.$$

The next theorem provides some insight into the problem of asymptotic pricing for complete models.

Theorem 4.10 *Under the following assumptions:*

- a) (NAA2) ,
 - b) the large market is complete, i.e. $\mathcal{Q}^n = \{Q^n\}$ is a singleton for each n ,
 - c) \mathbb{H} is bounded, i.e. $H_n \leq K$, for all n , where K is a positive constant,
- we have $v(\mathbb{H}) = \tilde{v}(\mathbb{H})$.*

Proof : First notice, that for any fixed $(A_n) \in \mathcal{A}_1$ by NAA2 we obtain

$$P^n(A_n) \rightarrow 1 \iff P^n(A_n^c) \rightarrow 0 \implies Q^n(A_n^c) \rightarrow 0.$$

Now consider two sequences:

$$x_n := \mathbf{E}^{Q^n}[H_n]$$

$$y_n := \mathbf{E}^{Q^n}[H_n \mathbf{1}_{A_n}].$$

The following holds:

$$x_n - y_n = \mathbf{E}^{Q^n}[H_n - H_n \mathbf{1}_{A_n}] = \mathbf{E}^{Q^n}[H_n \mathbf{1}_{A_n^c}] \leq K \cdot Q^n(A_n^c) \rightarrow 0$$

and thus

$$\lim_n x_n = \lim_n y_n.$$

Taking infimum over all $(A_n) \in \mathcal{A}_1$ we obtain the required result.

$$v(\mathbb{H}) = \lim_n x_n = \inf_{(A_n) \in \mathcal{A}} \lim_n y_n = \tilde{v}(\mathbb{H})$$

□

Remark 4.11 Assume that NAA2 holds. For incomplete market we can define the analogous sequences as in Theorem 4.10:

$$x_n := \sup_{Q \in \mathcal{Q}^n} \mathbf{E}^Q[H_n]$$

$$y_n := \sup_{Q \in \mathcal{Q}^n} \mathbf{E}^Q[H_n \mathbf{1}_{A_n}]$$

and for these sequences we obtain analogous inequality

$$x_n - y_n \leq \sup_{Q \in \mathcal{Q}^n} \mathbf{E}^Q[H_n - H_n \mathbf{1}_{A_n}] = \sup_{Q \in \mathcal{Q}^n} \mathbf{E}^Q[H_n \mathbf{1}_{A_n^c}] \leq K \cdot \bar{Q}^n(A_n^c).$$

However, we do not know if the last term goes to 0 as $n \rightarrow \infty$. We know that $\underline{Q}^n(A_n^c) \rightarrow 0$ only and this is insufficient to perform the above proof for incomplete markets.

5 The large Black-Scholes model

Let W_t^1, W_t^2, \dots be a sequence of independent standard Brownian motions on a filtered probability space $(\Omega, \mathcal{F}_t, \mathcal{F}, P), t \in [0, T]$. We will consider a stationary market, where the n -th small market has its natural filtration i.e. $\mathcal{F}_t^n = \sigma((W_s^1, \dots, W_s^n)_{s \in [0, t]})$ and $\mathcal{F}^n = \mathcal{F}_T^n$. The n -th objective measure is an adequate restriction of P i.e. $P^n = P|_{\mathcal{F}^n}$ and the discounted price processes are given by

$$dS_t^i = S_t^i(b_i dt + \sigma_i dW_t^i) \quad i = 1, 2, \dots, n, \quad t \in [0, T]$$

where $b_i \in \mathbb{R}, \sigma_i > 0$ for $i = 1, 2, \dots, n$. Such sequence forms a complete large market with martingale measures given by densities

$$\frac{dQ^n}{dP^n} = Z_n = e^{-(\theta^n, \mathbf{W}_T^n) - \frac{1}{2}\|\theta^n\|^2 T}$$

where $\theta^n = (\frac{b_1}{\sigma_1}, \dots, \frac{b_n}{\sigma_n})$ and $\mathbf{W}_t^n = (W_t^1, \dots, W_t^n)$. Recall, that $\mathbf{W}_t^{*n} = \mathbf{W}_t^n + \theta^n t$ is a Brownian motion under Q^n . In this setting we show more indirect proofs for the absence of asymptotic arbitrage using methods of mathematical statistics for searching optimal non-randomized tests (see Lemma 2.10). The shortcoming of this approach is that it works for deterministic coefficients only. In this section we show also, that Theorem 4.10 and Remark 4.9 remain true for random variables satisfying some integrability conditions, which are satisfied for widely used derivatives.

For this section use let us introduce a class of sequences (ε_n) which take values in the interval $[0, 1]$ and converging to 0. Such class will be denoted by \mathcal{E} .

Theorem 5.1 For $\varepsilon > 0$ let A_ε^n denote a solution of the problem

$$A \in \mathcal{F}^n : \begin{cases} P^n(A) \rightarrow \max \\ Q^n(A) \leq \varepsilon. \end{cases}$$

Then the following conditions are equivalent

- 1) NAA1
- 2) $(P^n) \triangleleft (Q^n)$

3) For any sequence $(\varepsilon_n) \in \mathcal{E}$, $P^n(A_{\varepsilon_n}^n) \rightarrow 0$ holds.

$$4) \sum_{i=1}^{\infty} \left(\frac{b_i}{\sigma_i}\right)^2 < \infty$$

Proof : Equivalence of (1) and (2) is proved in [8].

(2) \implies (3) Let (ε_n) be any element of \mathcal{E} . Then $Q^n(A_{\varepsilon_n}^n) \leq \varepsilon_n \rightarrow 0$ and thus by (2), $P^n(A_{\varepsilon_n}^n) \rightarrow 0$ holds.

(3) \implies (2) Let $A_n \in \mathcal{F}^n$ be s.t. $Q^n(A_n) \rightarrow 0$. Then $\varepsilon_n := Q^n(A_n)$ belongs to class \mathcal{E} and by (3), $P^n(A_n) \leq P^n(A_{\varepsilon_n}^n) \rightarrow 0$ holds.

(3) \iff (4) Statistical methods provide an explicit form of the set A_{ε}^n . According to the Neyman-Pearson Lemma 2.10 it is of the form $A_{\varepsilon}^n = \left\{ \frac{dP^n}{dQ^n} \geq \gamma \right\}$, where γ is a constant s.t. $Q^n(A_{\varepsilon}^n) = \varepsilon$. This construction provides

$$\begin{aligned} A_{\varepsilon}^n &= \left\{ e^{(\theta^n, \mathbf{W}_T^n) + \frac{1}{2} \|\theta^n\|^2 T} \geq \gamma \right\} = \left\{ (\theta^n, \mathbf{W}_T^n) \geq \ln \gamma - \frac{1}{2} \|\theta^n\|^2 T \right\} \\ &= \left\{ (\theta^n, (\mathbf{W}_T^{*n} - \theta T)) \geq \ln \gamma - \frac{1}{2} \|\theta^n\|^2 T \right\} = \left\{ (\theta^n, \mathbf{W}_T^{*n}) \geq \ln \gamma + \frac{1}{2} \|\theta^n\|^2 T \right\}. \end{aligned}$$

Solving the following equation:

$$Q^n(A_{\varepsilon}^n) = Q^n \left\{ (\theta^n, \mathbf{W}_T^{*n}) \geq \ln \gamma + \frac{1}{2} \|\theta^n\|^2 T \right\} = 1 - \Phi \left(\frac{\ln \gamma + \frac{1}{2} \|\theta^n\|^2 T}{\|\theta^n\| \sqrt{T}} \right) = \varepsilon$$

we obtain

$$\gamma = e^{\|\theta^n\| \sqrt{T} \Phi^{-1}(1-\varepsilon) - \frac{1}{2} \|\theta^n\|^2 T}.$$

We calculate the value $P^n(A_{\varepsilon}^n)$.

$$\begin{aligned} P^n(A_{\varepsilon}^n) &= P^n \left((\theta^n, \mathbf{W}_T^n) \geq \ln \gamma - \frac{1}{2} \|\theta^n\|^2 T \right) = 1 - \Phi \left(\frac{\ln \gamma - \frac{1}{2} \|\theta^n\|^2 T}{\|\theta^n\| \sqrt{T}} \right) \\ &= 1 - \Phi \left(\frac{\|\theta^n\| \sqrt{T} \Phi^{-1}(1-\varepsilon) - \frac{1}{2} \|\theta^n\|^2 T}{\|\theta^n\| \sqrt{T}} \right) = 1 - \Phi \left(\Phi^{-1}(1-\varepsilon) - \|\theta^n\| \sqrt{T} \right) \end{aligned}$$

Now observe that if $\sum_{i=1}^{\infty} \left(\frac{b_i}{\sigma_i}\right)^2 < \infty$ then for any $(\varepsilon_n) \in \mathcal{E}$

$$1 - \Phi \left(\Phi^{-1}(1-\varepsilon_n) - \|\theta^n\| \sqrt{T} \right) \rightarrow 0.$$

If $\sum_{i=1}^{\infty} \left(\frac{b_i}{\sigma_i}\right)^2 = \infty$ then $\varepsilon_n := 1 - \Phi(1 + \|\theta^n\| \sqrt{T}) \rightarrow 0$ and

$$1 - \Phi \left(\Phi^{-1}(1-\varepsilon_n) - \|\theta^n\| \sqrt{T} \right) = 1 - \Phi(1) \rightarrow 0.$$

□

The next two theorems provide characterization of NAA2, SAA1 and SAA2. The proofs are similar and therefore we sketch some parts of them only.

Theorem 5.2 For $\varepsilon > 0$ let A_{ε}^n denote a solution of the problem

$$A \in \mathcal{F}^n : \begin{cases} Q^n(A) \rightarrow \max \\ P^n(A) \leq \varepsilon. \end{cases}$$

Then the following conditions are equivalent

1) NAA2

2) $(Q^n) \triangleleft (P^n)$

3) For any sequence $(\varepsilon_n) \in \mathcal{E}$, $Q^n(A_{\varepsilon_n}^n) \rightarrow 0$ holds.

4) $\sum_{i=1}^{\infty} (\frac{b_i}{\sigma_i})^2 < \infty$

Proof : (3) \iff (4) The set A_{ε}^n is of the form

$$A_{\varepsilon}^n = \left\{ \frac{dQ^n}{dP^n} \geq \gamma \right\}$$

where γ is s.t. $P^n(A_{\varepsilon}^n) = \varepsilon$. This procedure yields

$$A_{\varepsilon}^n = \left\{ (\theta^n, \mathbf{W}_T^n) \leq \Phi \left(\frac{\ln \frac{1}{\gamma} - \frac{1}{2} \|\theta^n\|^2 T}{\|\theta^n\| \sqrt{T}} \right) \right\}$$

$$\gamma = e^{-[\Phi^{-1}(\varepsilon) \|\theta^n\| \sqrt{T} + \frac{1}{2} \|\theta^n\|^2 T]}$$

and $Q_n(A_{\varepsilon}^n) = \Phi \left(\Phi^{-1}(\varepsilon) + \|\theta^n\| \sqrt{T} \right)$.

If $\sum_{i=1}^{\infty} (\frac{b_i}{\sigma_i})^2 < \infty$ then for any $(\varepsilon_n) \in \mathcal{E}$

$$\Phi \left(\Phi^{-1}(\varepsilon_n) + \|\theta^n\| \sqrt{T} \right) \rightarrow 0.$$

If $\sum_{i=1}^{\infty} (\frac{b_i}{\sigma_i})^2 = \infty$ then taking $\varepsilon_n := \Phi(1 - \|\theta^n\| \sqrt{T}) \rightarrow 0$ we obtain

$$\Phi \left(\Phi^{-1}(\varepsilon_n) + \|\theta^n\| \sqrt{T} \right) = \Phi(1) \not\rightarrow 0.$$

□

Theorem 5.3 For $\varepsilon > 0$ let A_{ε}^n denote a solution of the problem

$$A \in \mathcal{F}^n : \begin{cases} P^n(A) \rightarrow \max \\ Q^n(A) \leq \varepsilon. \end{cases}$$

Then the following conditions are equivalent

1) SAA1

2) SAA2

3) $P^n \triangle Q^n$

4) $Q^n \triangle P^n$

5) There exists a sequence $(\varepsilon_n) \in \mathcal{E}$ s.t. $P^n(A_{\varepsilon_n}^n) \rightarrow 1$

6) $\sum_{i=1}^{\infty} (\frac{b_i}{\sigma_i})^2 = \infty$.

Notice, that the conditions for the set A_ε^n are based on property $P^n \Delta Q^n$. One can base the proof on the property $Q^n \Delta P^n$. This requires replacing measures P^n and Q^n in the conditions for A_ε^n . The first four conditions are proved in [8] and are included in the formulation above for the clarity of exposition only. Equivalence of (3) and (5) are easy to prove.

Proof : (5) \iff (6) We use the construction of A_ε^n found in the proof of Th. 5.1

$$A_\varepsilon^n = \left\{ (\theta^n, \mathbf{W}_T^{*n}) \geq \ln \gamma + \frac{1}{2} \|\theta^n\|^2 T \right\}$$

$$\gamma = e^{\|\theta^n\| \sqrt{T} \Phi^{-1}(1-\varepsilon) - \frac{1}{2} \|\theta^n\|^2 T}$$

$$P^n(A_\varepsilon^n) = 1 - \Phi \left(\Phi^{-1}(1-\varepsilon) - \|\theta^n\| \sqrt{T} \right)$$

If $\sum_{i=1}^\infty (\frac{b_i}{\sigma_i})^2 < \infty$ then for any $(\varepsilon_n) \in \mathcal{E}$, $1 - \Phi \left(\Phi^{-1}(1-\varepsilon_n) - \|\theta^n\| \sqrt{T} \right) \rightarrow 0$ holds. If $\sum_{i=1}^\infty (\frac{b_i}{\sigma_i})^2 = \infty$ then $\varepsilon_n := 1 - \Phi(\frac{1}{2} \|\theta^n\| \sqrt{T}) \rightarrow 0$ and $1 - \Phi \left(\Phi^{-1}(1-\varepsilon_n) - \|\theta^n\| \sqrt{T} \right) \rightarrow 1$.

□

In the sequel we will characterize the weak price of \mathbb{H} satisfying some integrability conditions. If $\mathbb{H} = H$, where H is one fixed random variable measurable with respect to \mathcal{F}^1 , then it is clear that $\mathbf{E}^{Q^n}[H]$ does not depend on n and thus indicates the strong price. This means that the investor doesn't have any profits from the fact that the market is getting large and that he can use greater and grater number of strategies. It turns out that he can not make any profits unless he uses 1-quantile hedging strategies. In this case, but if $\sum_{i=1}^\infty (\frac{b_i}{\sigma_i})^2 = \infty$, the initial endowment can be reduced to 0, i.e. the weak price is equal to 0. The condition $\sum_{i=1}^\infty (\frac{b_i}{\sigma_i})^2 < \infty$ guaranties that the investor is not able to make any profits at all, no matter what strategies he uses, because then $v(\mathbb{H}) = \tilde{v}(\mathbb{H})$.

Theorem 5.4 *Let \mathbb{H} be a contingent claim on a large Black-Scholes market with constant coefficients. Then*

- 1) if $\sum_{i=1}^\infty (\frac{b_i}{\sigma_i})^2 < \infty$ and $\overline{\lim}_n \mathbf{E}[H_n^{1+\delta}] < \infty$ for some $\delta > 0$ then $\tilde{v}(\mathbb{H}) = v(\mathbb{H})$.
- 2) if $\sum_{i=1}^\infty (\frac{b_i}{\sigma_i})^2 = \infty$ and $\overline{\lim}_n \mathbf{E}[H_n^{4+\delta}] < \infty$ for some $\delta > 0$ then $\tilde{v}(\mathbb{H}) = 0$.

Proof : (1) For any sequence $(A_n) \in \mathcal{A}_1$ define $x_n := \mathbf{E}^{Q^n}[H_n]$, $y_n := \mathbf{E}^{Q^n}[H_n \mathbf{1}_{A_n}]$. Let $p, q, p', q' > 1$ be real numbers and s.t. $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{p'} + \frac{1}{q'} = 1$. Using Hölder inequality twice to the difference $x_n - y_n$ we obtain:

$$\begin{aligned} x_n - y_n &= \mathbf{E}^{Q^n}[H_n \mathbf{1}_{A_n^c}] = \mathbf{E}[Z_n H_n \mathbf{1}_{A_n^c}] \leq \left(\mathbf{E}(Z_n H_n)^p \right)^{\frac{1}{p}} \left(P(A_n^c) \right)^{\frac{1}{q}} \\ &\leq \left(\left(\mathbf{E}(Z_n^{pp'}) \right)^{\frac{1}{p'}} \left(\mathbf{E}(H_n^{pq'}) \right)^{\frac{1}{q'}} \right)^{\frac{1}{p}} \left(P(A_n^c) \right)^{\frac{1}{q}} \\ &= \left(\mathbf{E}(Z_n^{pp'}) \right)^{\frac{1}{pp'}} \left(\mathbf{E}(H_n^{pq'}) \right)^{\frac{1}{pq'}} \left(P(A_n^c) \right)^{\frac{1}{q}}. \end{aligned}$$

Straightforward calculations yields

$$\left(\mathbf{E}(Z_n^{pp'}) \right)^{\frac{1}{pp'}} = e^{\frac{1}{2} \|\theta^n\|^2 T (pp' - 1)} \quad (5.4.1)$$

and thus NAA2 guaranties that $\lim_{n \rightarrow \infty} \left(\mathbf{E}(Z_n^{pp'}) \right)^{\frac{1}{pp'}} < \infty$. Taking p, p' s.t. $pp' = 1 + \delta$ and using fact that $\lim_{n \rightarrow \infty} \left(P(A_n^c) \right)^{\frac{1}{q}} = 0$ we conclude that $\lim_{n \rightarrow \infty} (x_n - y_n) = 0$. Thus $\underline{\lim} x_n = \underline{\lim} y_n$ and taking infimum over all $(A_n) \in \mathcal{A}_1$ we get $\tilde{v}(\mathbb{H}) = v(\mathbb{H})$.

(2) For any $(A_n) \in \mathcal{A}_1$, $p, p' > 1$ and q, q' s.t. $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{p'} + \frac{1}{q'} = 1$ using Hölder inequalities we obtain:

$$\begin{aligned} \mathbf{E}^{Q^n}[H_n \mathbf{1}_{A_n}] &\leq \left(\mathbf{E}^{Q^n}(H_n^p) \right)^{\frac{1}{p}} \left(Q^n(A_n) \right)^{\frac{1}{q}} = \left(\mathbf{E}(Z_n H_n^p) \right)^{\frac{1}{p}} \left(Q^n(A_n) \right)^{\frac{1}{q}} \\ &\leq \left(\left(\mathbf{E} Z_n^{p'} \right)^{\frac{1}{p'}} \left(\mathbf{E} H_n^{pq'} \right)^{\frac{1}{q'}} \right)^{\frac{1}{p}} \left(Q^n(A_n) \right)^{\frac{1}{q}} = \left(\mathbf{E} Z_n^{p'} \right)^{\frac{1}{pp'}} \left(\mathbf{E} H_n^{pq'} \right)^{\frac{1}{pq'}} \left(Q^n(A_n) \right)^{\frac{1}{q}} \end{aligned}$$

Now, similarly to the previously used methods let us solve an auxiliary problem of finding set A_ε^n s.t.

$$A \in \mathcal{F}^n : \begin{cases} Q^n(A) \longrightarrow \min \\ P^n(A) \geq 1 - \varepsilon. \end{cases}$$

Analogous calculations provide:

$$A_\varepsilon^n = \left\{ \frac{dQ^n}{dP^n} \leq \gamma \right\} = \left\{ (\theta^n, \mathbf{W}^n) \geq -\ln \gamma \frac{1}{2} \|\theta^n\|^2 T \right\}$$

$$\gamma = e^{-[\Phi^{-1}(\varepsilon) \|\theta^n\| \sqrt{T} + \frac{1}{2} \|\theta^n\|^2 T]}$$

$$Q^n(A_\varepsilon^n) = \Phi \left(-\Phi^{-1}(\varepsilon) - \|\theta^n\| \sqrt{T} \right).$$

Taking $p = 2 + \frac{1}{2}\delta$, $p' = 2$, $\varepsilon_n = \Phi \left(-\ln(\|\theta^n\| \sqrt{T}) \right)$ (AA2 guaranties that $\varepsilon_n \rightarrow 0$) we get

$$\overline{\lim} \mathbf{E}[H_n^{pq'}] = \overline{\lim} \mathbf{E}[H_n^{4+\delta}] < \infty$$

and

$$\begin{aligned} \left(\left(\mathbf{E} Z_n^{p'} \right)^{\frac{1}{pp'}} \left(Q^n(A_{\varepsilon_n}^n) \right)^{\frac{1}{q}} \right)^q &= e^{\frac{1}{2} \frac{p'-1}{p-1} \|\theta^n\|^2 T} \Phi \left(-\Phi^{-1}(\varepsilon_n) - \|\theta^n\| \sqrt{T} \right) \\ &= e^{\frac{1}{2+\delta} \|\theta^n\|^2 T} \Phi \left(\ln(\|\theta^n\| \sqrt{T}) - \|\theta^n\| \sqrt{T} \right) \end{aligned} \quad (5.4.2)$$

Replacing $\|\theta^n\| \sqrt{T}$ by x for the sake of convenience, we calculate the following limit using d'Hospital formula.

$$\begin{aligned} \lim_{x \rightarrow \infty} e^{\frac{1}{2+\delta} x^2} \Phi(\ln x - x) &= \lim_{x \rightarrow \infty} \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\ln x - x)^2} \left(\frac{1}{x} - 1 \right)}{e^{-\frac{1}{2+\delta} x^2} \left(-\frac{1}{2+\delta} \right) 2x} \\ &= \lim_{x \rightarrow \infty} -\frac{2+\delta}{2\sqrt{2\pi}} \left[\frac{e^{x^2(\frac{1}{2+\delta} - \frac{1}{2}) - \frac{1}{2} \ln^2 x + x \ln x}}{x^2} - \frac{e^{x^2(\frac{1}{2+\delta} - \frac{1}{2}) - \frac{1}{2} \ln^2 x + x \ln x}}{x} \right] = 0 \end{aligned}$$

The limit is equal to 0 since: $\lim x^2(\frac{1}{2+\delta} - \frac{1}{2}) - \frac{1}{2} \ln^2 x + x \ln x = -\infty$.

Summarizing, we have shown that $\lim_{n \rightarrow \infty} \left(\mathbf{E} Z_n^{p'} \right)^{\frac{1}{pp'}} \left(\mathbf{E} H^{pq'} \right)^{\frac{1}{pq'}} \left(Q^n(A_{\varepsilon_n}^n) \right)^{\frac{1}{q}} = 0$ for the adequate parameters and thus $\tilde{v}(\mathbb{H}) = 0$. \square

Remark 5.5 *The integrability conditions imposed on \mathbb{H} in the second item of Theorem 5.4 can be a little bit weakened. It follows from 5.4.2 that we have to find parameters $p, p' > 1$ s.t. $\frac{1}{2} \frac{p'-1}{p-1} = \frac{1}{2+\delta}$. We can impose additional requirement: $pq' \rightarrow \min$. Then it can be checked, that the solution is: $p = 1 + \sqrt{\frac{2+\delta}{2}}$, $p' = \frac{2+2\sqrt{\frac{2+\delta}{2}}+\delta}{2+\delta}$ and $pq' = \frac{\sqrt{2}}{2} + \frac{1}{2}(4+\delta) + \frac{\sqrt{2}}{2}\sqrt{2+\delta}$. If $\delta \rightarrow 0$ then pq' is arbitrarily close to $\frac{\sqrt{2}}{2} + 2 + 1 < 4$. Thus, we can assume that*

$$\lim_n \mathbf{E}[H_n^{\frac{\sqrt{2}}{2} + \frac{1}{2}(4+\delta) + \frac{\sqrt{2}}{2}\sqrt{2+\delta}}] < \infty \text{ for some } \delta > 0.$$

The next theorem provides a more precise characterization of the α -quantile price. But first let us impose a regularity assumption on the random variables $H_n Z_n$.

Assumption 5.6 *The random variable $H_n Z_n$ has a continuous distribution function with respect to the measure P^n .*

By $q_n(\alpha)$ we denote the α -quantile of $H_n Z_n$, i.e. $q_n(\alpha) = \{\inf x : P^n(H_n Z_n \leq x) \geq \alpha\}$.

Denote by \mathcal{B}_α a set of sequences satisfying

$$\lim_{n \rightarrow \infty} \beta_n \geq \alpha.$$

Theorem 5.7 *Let \mathbb{H} be a contingent claim on a large Black-Scholes model with constant coefficients.*

1) *Under assumption 5.6 the α -quantile price is given by the formula*

$$v_\alpha(\mathbb{H}) = \inf_{(\beta_n) \in \mathcal{B}_\alpha} \lim_{n \rightarrow \infty} \mathbf{E} \left[H_n Z_n \mathbf{1}_{\{H_n Z_n \leq q_n(\beta_n)\}} \right].$$

2) *Let assumption 5.6 be satisfied. If $\lim_n \mathbf{E}[H_n^{1+\delta}] < \infty$ for some $\delta > 0$ and $\sum_{i=1}^\infty (\frac{b_i}{\sigma_i})^2 < \infty$ then*

$$v_\alpha(\mathbb{H}) = \lim_{n \rightarrow \infty} \mathbf{E} \left[H_n Z_n \mathbf{1}_{\{H_n Z_n \leq q_n(\alpha)\}} \right].$$

Moreover, $v_\alpha(\mathbb{H})$ is a Lipschitz, increasing function of α taking values in the interval $[0, v(\mathbb{H})]$.

3) *If $\lim_n \mathbf{E}[H_n^{4+\delta}] < \infty$ for some $\delta > 0$ and $\sum_{i=1}^\infty (\frac{b_i}{\sigma_i})^2 = \infty$ then $v_\alpha(\mathbb{H}) = 0$ for each $\alpha \in [0, 1]$.*

Proof: (1) By Theorem 4.5 the α -quantile price is given by the formula:

$$v_\alpha(\mathbb{H}) = \inf_{(A_n) \in \mathcal{A}_\alpha} \lim_n \sup_{Q \in \mathcal{Q}^n} \mathbf{E}^Q[H_n \mathbf{1}_{A_n}].$$

Let us consider any $(A_n) \in \mathcal{A}_\alpha$ and define $\beta_n := P^n(A_n)$. Denote by \tilde{A}_n a solution of the following problem:

$$\tilde{A}_n : \begin{cases} \mathbf{E}^{Q^n}[H_n \mathbf{1}_{A_n}] \longrightarrow \min \\ P^n(A_n) \geq \beta_n. \end{cases}$$

If we introduce measure \tilde{Q}^n by the density $\frac{d\tilde{Q}^n}{dQ^n} := \frac{H_n}{\mathbf{E}^{Q^n}[H_n]}$, then the above problem can be written in the equivalent form:

$$\tilde{A}_n : \begin{cases} \tilde{Q}^n(A_n) \longrightarrow \min \\ P^n(A_n) \geq \beta_n. \end{cases}$$

Therefore by Lemma 2.10 we conclude that \tilde{A}_n is of the form: $\{H_n Z_n \leq \gamma\}$, where γ is a constant s.t. $P^n(H_n Z_n \leq \gamma) = \beta_n$. By Assumption 5.6 we know that there exists such γ and it is equal to $q_n(\beta_n)$. Thus $\tilde{A}_n = \{H_n Z_n \leq q_n(\beta_n)\}$ and

$$\mathbf{E}^{Q^n}[H_n \mathbf{1}_{A_n}] \geq \mathbf{E}^{Q^n}[H_n \mathbf{1}_{\{H_n Z_n \leq q_n(\beta_n)\}}].$$

Letting $n \rightarrow \infty$ and taking infimum over all $(A_n) \in \mathcal{A}_\alpha$ we obtain:

$$v_\alpha(\mathbb{H}) \geq \inf_{(\beta_n) \in \mathcal{B}_\alpha} \lim_{n \rightarrow \infty} \mathbf{E}[H_n Z_n \mathbf{1}_{\{H_n Z_n \leq q_n(\beta_n)\}}]. \quad (5.7.3)$$

However, $P^n(H_n Z_n \leq q_n(\beta_n)) = \beta_n$, so $\{H_n Z_n \leq q_n(\beta_n)\} \in \mathcal{A}_\alpha$ and this implies equality in 5.7.3.

(2) Let $\alpha, \beta \in [0, 1]$ be two real numbers s.t. $\beta < \alpha$. For $p, q, p', q' > 1$ s.t. $\frac{1}{p} + \frac{1}{q} = 1, \frac{1}{p'} + \frac{1}{q'} = 1$ we have the following inequality:

$$\begin{aligned} & \mathbf{E}[H_n Z_n \mathbf{1}_{\{H_n Z_n \leq q_n(\alpha)\}}] - \mathbf{E}[H_n Z_n \mathbf{1}_{\{H_n Z_n \leq q_n(\beta)\}}] = \mathbf{E}[H_n Z_n \mathbf{1}_{\{q_n(\beta) \leq H_n Z_n \leq q_n(\alpha)\}}] \\ & \leq \left(\mathbf{E}(Z_n^{pp'}) \right)^{\frac{1}{pp'}} \left(\mathbf{E}(H_n^{pq'}) \right)^{\frac{1}{pq'}} \left(P(q_n(\beta) \leq H_n Z_n \leq q_n(\alpha)) \right)^{\frac{1}{q}} = \left(\mathbf{E}(Z_n^{pp'}) \right)^{\frac{1}{pp'}} \left(\mathbf{E}(H_n^{pq'}) \right)^{\frac{1}{pq'}} (\alpha - \beta) \end{aligned}$$

However, by 5.4.1 we have $\left(\mathbf{E}(Z_n^{pp'}) \right)^{\frac{1}{pp'}} \leq \lim_{n \rightarrow \infty} \left(\mathbf{E}(Z_n^{pp'}) \right)^{\frac{1}{pp'}} < \infty$. Taking p, q' s.t. $pq' = 1 + \delta$ and denoting $K_1 := \lim_{n \rightarrow \infty} \left(\mathbf{E}(Z_n^{pp'}) \right)^{\frac{1}{pp'}}$ and $K_2 := \left(\mathbf{E}(H_n^{pq'}) \right)^{\frac{1}{pq'}}$, we obtain

$$\mathbf{E}[H_n Z_n \mathbf{1}_{\{H_n Z_n \leq q_n(\alpha)\}}] - \mathbf{E}[H_n Z_n \mathbf{1}_{\{H_n Z_n \leq q_n(\beta)\}}] \leq K_1 K_2 (\alpha - \beta).$$

and interchanging the role of α and β we obtain

$$| \mathbf{E}[H_n Z_n \mathbf{1}_{\{H_n Z_n \leq q_n(\alpha)\}}] - \mathbf{E}[H_n Z_n \mathbf{1}_{\{H_n Z_n \leq q_n(\beta)\}}] | \leq K_1 K_2 | \alpha - \beta |. \quad (5.7.4)$$

Now consider $(\beta_n) \in \mathcal{B}_\alpha$. If $\lim_{n \rightarrow \infty} \beta_n > \alpha$ then $\mathbf{E}[H_n Z_n \mathbf{1}_{\{H_n Z_n \leq q_n(\beta_n)\}}] > \mathbf{E}[H_n Z_n \mathbf{1}_{\{H_n Z_n \leq q_n(\alpha)\}}]$ and thus $\lim_{n \rightarrow \infty} \mathbf{E}[H_n Z_n \mathbf{1}_{\{H_n Z_n \leq q_n(\beta_n)\}}] \geq \lim_{n \rightarrow \infty} \mathbf{E}[H_n Z_n \mathbf{1}_{\{H_n Z_n \leq q_n(\alpha)\}}]$. If $\lim_{n \rightarrow \infty} \beta_n = \alpha$ then by 5.7.4 we have $| \mathbf{E}[H_n Z_n \mathbf{1}_{\{H_n Z_n \leq q_n(\alpha)\}}] - \mathbf{E}[H_n Z_n \mathbf{1}_{\{H_n Z_n \leq q_n(\beta_n)\}}] | \leq K_1 K_2 | \alpha - \beta_n |$ and letting $n \rightarrow \infty$ we obtain $\lim_{n \rightarrow \infty} \mathbf{E}[H_n Z_n \mathbf{1}_{\{H_n Z_n \leq q_n(\beta_n)\}}] = \lim_{n \rightarrow \infty} \mathbf{E}[H_n Z_n \mathbf{1}_{\{H_n Z_n \leq q_n(\alpha)\}}]$. The conclusion from these two cases is that $v_\alpha(\mathbb{H}) \geq \lim_{n \rightarrow \infty} \mathbf{E}[H_n Z_n \mathbf{1}_{\{H_n Z_n \leq q_n(\alpha)\}}]$. However, $\{H_n Z_n \leq q_n(\alpha)\} \in \mathcal{A}_\alpha$ and therefore

$$v_\alpha(\mathbb{H}) = \lim_{n \rightarrow \infty} \mathbf{E}[H_n Z_n \mathbf{1}_{\{H_n Z_n \leq q_n(\alpha)\}}]. \quad (5.7.5)$$

Letting $n \rightarrow \infty$ in 5.7.4 and using 5.7.5 we obtain:

$$|v_\alpha(\mathbb{H}) - v_\beta(\mathbb{H})| \leq K_1 K_2 |\alpha - \beta|,$$

which proves that $v_\alpha(\mathbb{H})$ is Lipschitz. It is clear by 5.7.5 that $v_\alpha(\mathbb{H})$ is increasing and that $v_0(\mathbb{H}) = 0$. By Theorem 5.4 $v_1(\mathbb{H}) = v(\mathbb{H})$.

(3) It is an immediate consequence of Theorem 5.4 (2), since $v_\alpha(\mathbb{H}) \leq \tilde{v}(\mathbb{H})$. \square

Remark 5.8 Consider the prices of a call option, i.e. $\mathbb{H} \equiv (S_T^1 - K)^+$. The distribution of $(S_T^1 - K)^+ Z_n$ is discontinuous in 0. Let $\alpha_0 := P^n((S_T^1 - K)^+ = 0)$. It is clear, that for $\alpha \leq \alpha_0$, $v_\alpha(\mathbb{H}) = 0$ holds. On the interval $(0, \infty)$ the distribution function is continuous, thus for $\alpha > \alpha_0$ Theorem 5.7 can be applied.

Conclusion

In this paper we have introduced and characterized two types of asymptotic prices. They are based on different treating of hedging risk which disappears in infinity. Relations between them strictly depend on the asymptotic arbitrage on the market. In case of the large Black-Scholes model with constant coefficients it was possible to find more indirect formula for the α -quantile price and state some properties of it. On this market there are two situations possible:

- 1) there is no asymptotic arbitrage of any kind - then the strong and the weak price are equal
- 2) there is asymptotic arbitrage of all kinds - then the weak price is equal to zero, while the strong is not.

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